First

PROGRESS REPORT

(Project SR-130)

on

RAPID PROPAGATION OF A CRACK IN A BRITTLE MATERIAL

by

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RAPID PROPAGATION OF A CRACK IN A BRITTLE MATERIAL[1]

by

Max J. Schilhansl[2]

1. The slow propagation of a crack has been theoretically examined by A. A. Griffith (1) (2), A. Smekal (3), K. Wolf (4) and L. Prandtl (5). The rapid propagation of a crack has been theoretically examined by E. H. Yoffe (6), who lists further references. The investigation is based on the assumption already made by A. A. Griffith that the initial crack can be considered as a slit in an infinite plate; the slit has the shape of an ellipse, the shorter semi-axis of which is parallel to the direction of the stress at infinity and approaches zero. Furthermore, it is assumed that this slit does not change its length when moving with the velocity v in the general direction of the longer semi-axis. With both assumptions, it has been found, that the crack propagates in a direction normal to the maximum tensile stress up to a critical velocity $v = 0.6v_{SH}$ -- where v_{SH} designates the velocity of propagation of shear waves -- at which the crack tends to become curved. This trend is confirmed by experiments according to a statement of E. H. Yoffe.

H. R. H. Schardin (7) found by experiments that a crack propagates with constant velocity if suddenly a load of very short duration ($\frac{1}{50,000}$ sec) is applied to the test specimen. A. Smekal (8)(9)(10) concluded from experiments that the propagation of a crack immediately after the starting of the crack cannot be examined by means of the mechanics of the continuum

[1]The results presented in this paper were obtained in the course of research sponsored by the Department of the Navy, Bureau of Ships, under Contract No. NObs 65917.

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[3]Numbers in parentheses refer to the Bibliography at the end of the paper.

but must be explained as a thermal effect. When the crack has reached a certain length, the velocity of the propagation of the crack grows very rapidly up to approximately 60% of the velocity of propagation of the shear waves and thereafter remains constant.

In the papers of A. Smekal (9)(10) further references are listed.

2. In the following pages, an attempt will be described to apply the mechanics of the continuum to the problem of the propagation of a crack in a plate being in a one-dimensional state of stress by static external loads. Of course, it is necessary to assume that the crack has already reached the point where the rapid increase of velocity of propagation begins.

For this purpose, the concept of L. Prandtl (5) will be used. Fig. 1 shows the plate and the system of coordinates to be employed. It is assumed that the initial crack is right at the center of the plate and that it consists of a slit perpendicular to the direction of the external load p; the length of the crack is $2\ell(t)$ at the time t. The actual plate is replaced by a model consisting of two beams that are connected by strings perpendicular to the axes of the beams from $x = \pm\ell$ to the edges of the plate. The length of these strings is equal to the gap of the initial crack so long as no load is applied.

This corresponds exactly to the concept of Prandtl (5). Moreover, a second system of strings should be assumed perpendicular to the first ones so that our concept comes closer to the concept of an isotropic material.

A system of coordinates x,y or x,w is used, the x-axis of which coincides with the axis of the beam and the y-axis or w-axis, respectively, passes through the middle of the crack. The symbol w denotes the deflection of the axis of the beam at any time t while the symbol y

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Max J. Schilhansl
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under

Department of the Navy
Bureau of Ships NObs-65917
BuShips Project No. NS-731-034

for

SHIP STRUCTURE COMMITTEE

SHIP STRUCTURE COMMITTEE

MEMBER AGENCIES:

BUREAU OF SHIPS, DEPT. OF NAVY
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ADDRESS CORRESPONDENCE TO:

SECRETARY
SHIP STRUCTURE COMMITTEE
U. S. COAST GUARD HEADQUARTERS
WASHINGTON 25, D. C.

March 25, 1955

Dear Sir:

As part of its research program related to the improvement of hull structures of ships, the Ship Structure Committee is sponsoring an investigation of Brittle Fracture Mechanics at Brown University. Herewith is a copy of the First Progress Report, SSC-87, of the investigation entitled "Rapid Propagation of a Crack in a Brittle Material," by Max J. Schilhansl.

The project is being conducted with the advisory assistance of the Committee on Ship Structural Design of the National Academy of Sciences-National Research Council.

Comments concerning this report are solicited and should be addressed to the Secretary, Ship Structure Committee.

This report is being distributed to those individuals and agencies associated with and interested in the work of the Ship Structure Committee.

Yours sincerely,

K. K. Cowart
K. K. COWART
Rear Admiral, U. S. Coast Guard
Chairman, Ship Structure Committee

is used for quantities which are independent of the time as, for instance, the amplitudes of an oscillatory motion in case of vibrations.

3. The following notation will be used:

E = modulus of elasticity in tension

G = modulus of elasticity in shear $(G = \frac{E}{2(1+\nu)})$

ν = Poisson's ratio

F = area of the cross section of the beam

F_s = equivalent area of the cross section in shear

ρ = mass density

$\mu = \rho F$ mass per unit length

I = moment of inertia of the cross sectional area

The differential equation of the elastic line of a beam reads

$$EI \frac{\partial^4 w}{\partial x^4} - \left(\mu \frac{EI}{GF_s} + \rho I\right) \frac{\partial^4 w}{\partial x^2 \partial t^2} + \mu \frac{\rho I}{GF_s} \frac{\partial^4 w}{\partial t^4} + \mu \frac{\partial^2 w}{\partial t^2} = 0. \tag{3.1}$$

so long as the beam is not supported by an elastic foundation and carries no external static loads. It is assumed here that there are no masses attached to the beam which would contribute to the mass inertia but not to the stiffness.

By equating the rotational mass moment of inertia ρI to zero, a simplified equation can be obtained

$$EI \frac{\partial^4 w}{\partial x^4} - \mu \frac{EI}{GF_s} \frac{\partial^4 w}{\partial x^2 \partial t^2} + \mu \frac{\partial^2 w}{\partial t^2} = 0. \tag{3.2}$$

on the other hand, if the shear deformation is neglected by equating $GF_s = \infty$, we shall have another simplified equation

$$EI \frac{\partial^4 w}{\partial x^4} - \rho I \frac{\partial^4 w}{\partial x^2 \partial t^2} + \mu \frac{\partial^2 w}{\partial t^2} = 0. \tag{3.3}$$

Next, we assume that a periodical disturbance $y_0 \sin \omega t$ is introduced at the point $x = 0$. In this case the solution of the above three equations is of the form

$$w(x,t) = -y_0 \sin\omega\left(\frac{x}{v} - t\right) \tag{3.4}$$

where v is the velocity of wave propagation.

Substitution of the solution (3.4) into Eq. (3.2) yields

$$\frac{1}{v^2} = \frac{1}{2}\frac{\mu}{GF_s} \pm \sqrt{\left(\frac{1}{2}\frac{\mu}{GF_s}\right)^2 + \frac{1}{\omega^2}\frac{\mu}{EI}} \tag{3.5}$$

The velocity v has a maximum for $\omega = \infty$ which is given by

$$v_{SH} = \sqrt{\frac{GF_s}{\mu}} = \sqrt{\frac{E}{\rho}\frac{1}{2(1+\nu)}\frac{F_s}{F}} \tag{3.6}$$

Substitution of the solution (3.4) into equation (3.3) yields

$$\frac{1}{v^2} = \frac{1}{2}\frac{\rho}{E} \pm \sqrt{\left(\frac{1}{2}\frac{\rho}{E}\right)^2 + \frac{1}{\omega^2}\frac{\mu}{EI}} \tag{3.7}$$

Again, there is a maximum velocity for $\omega = \infty$ which is

$$v_L = \sqrt{\frac{E}{\rho}} \tag{3.8}$$

v_{SH} and v_L are the velocities of propagation of shear waves and of longitudinal waves, respectively.

Substitution of the solution (3.4) into the Eq. (3.1) yields:

$$\frac{1}{v^2} = \frac{1}{2}\left(\frac{\mu}{GF_s} + \frac{\rho}{E}\right) \pm \sqrt{\left(\frac{1}{2}\frac{\mu}{GF_s} + \frac{1}{2}\frac{\rho}{E}\right)^2 - \frac{\mu}{GF_s}\frac{\rho}{E} + \frac{1}{\omega^2}\frac{\mu}{EI}} \tag{3.9}$$

In view of Eqs. (3.5) and (3.7), Eq. (3.8) can be written, if $\omega = \infty$

$$\frac{1}{v^2} = \left(\frac{1}{2v_{SH}^2} + \frac{1}{2v_L^2}\right)\left[1 \pm \sqrt{1 - 4\frac{\mu}{GF_s}\frac{\rho}{E}\frac{1}{\left(\frac{\mu}{GF_s} + \frac{\rho}{E}\right)^2}}\right] \tag{3.10}$$

With $G = \frac{E}{2(1+\nu)}$ and abbreviating the ratio F/F_s by the symbol φ, we find

$$\frac{\mu}{GF_s}\frac{\rho}{E}\frac{1}{\left(\frac{\mu}{GF_s}+\frac{\rho}{E}\right)^2} = \frac{2\varphi(1+\nu)}{(2\varphi(1+\nu)+1)^2} \tag{3.11}$$

Thence with $\omega = \infty$

$$\frac{1}{v^2} = \frac{\rho}{2E}\left\{[2\varphi(1+\nu)+1] \pm 2\varphi(1+\nu) - 1]\right\} \tag{3.12}$$

With the use of the upper sign in the parenthesis, it follows that

$$\frac{1}{v_1^{\ 2}} = \frac{1}{v_{SH}^{\ 2}} \tag{3.13}$$

while with the use of the lower sign in the parenthesis, it follows that

$$\frac{1}{v_2^{\ 2}} = \frac{1}{v_L^{\ 2}} \tag{3.14}$$

4. If the beam is connected with a second beam as shown in Fig. 1 by elastic strings of the elastic constant c, the external load $cwdx$ must be added to the d'Alembert force $\mu \frac{\partial^2 w}{\partial t^2} dx$. The shear force Q at a certain abscissa x increases by the amount $\frac{\partial Q}{\partial x} dx$ as we proceed from the abscissa x to the abscissa $x + dx$. Consequently

$$\frac{\partial Q}{\partial x} dx = \mu \frac{\partial^2 w}{\partial t^2} dx + cwdx \tag{4.1}$$

Equilibrium of the element dx is established if

$$\frac{\partial M}{\partial x} = Q \tag{4.2}$$

Let the angular rotation of a cross section be denoted by ψ. So long as the deformation in shear is neglected, ψ can be put equal to $\frac{\partial w}{\partial x}$.
If the shear deformation is taken into account, the angle ψ is no longer equal to $\frac{\partial w}{\partial x}$; thus, the stress-strain relations are

$$M = -EI \frac{\partial \psi}{\partial x} \tag{4.3}$$

and

$$Q = GF_s \left(\frac{\partial w}{\partial x} - \psi\right). \tag{4.4}$$

Differentiating Eq. (4.4) with respect to x yields

$$\frac{\partial Q}{\partial x} = GF_s \left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \psi}{\partial x}\right). \tag{4.5}$$

Introducing Eq. (4.5) into Eq. (4.1), we obtain

$$\mu \frac{\partial^2 w}{\partial t^2} + cw = GF_s \left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \psi}{\partial x}\right). \tag{4.6}$$

The solution of Eq. (4.6) for $\frac{\partial \psi}{\partial x}$ gives

$$\frac{\partial \psi}{\partial x} = \frac{\partial^2 w}{\partial x^2} - \frac{\mu}{GF_s} \frac{\partial^2 w}{\partial t^2} - \frac{c}{GF_s} w. \tag{4.7}$$

Differentiating Eq. (4.3) with respect to x, substituting the result into Eq. (4.2) and, finally, comparing with Eq. (4.4), we find

$$-EI \frac{\partial^2 \psi}{\partial x^2} = GF_s \left(\frac{\partial w}{\partial x} - \psi\right). \tag{4.8}$$

Elimination of the variable ψ and its derivatives from Eqs. (4.7) and (4.8) yields, finally,

$$-EI \left(\frac{\partial^4 w}{\partial x^4} - \frac{\mu}{GF_s} \frac{\partial^4 w}{\partial x^2 \partial t^2} - \frac{c}{GF_s} \frac{\partial^2 w}{\partial x^2}\right) = \mu \frac{\partial^2 w}{\partial t^2} + cw \tag{4.9}$$

or

$$EI \frac{\partial^4 w}{\partial x^4} - \mu \frac{EI}{GF_s} \frac{\partial^4 w}{\partial x^2 \partial t^2} + \mu \frac{\partial^2 w}{\partial t^2} + c\left(w - \frac{EI}{GF_s} \frac{\partial^2 w}{\partial x^2}\right) = 0 \tag{4.10}$$

If it is desired to also take the rotational inertia into account, Eq. (4.2) must be replaced by

$$\frac{\partial M}{\partial x} = Q - \rho I \frac{\partial^2 \psi}{\partial t^2} \tag{4.11}$$

By proceeding as previously, the differential equation of the elastic line is obtained as

$$\begin{aligned} EI \frac{\partial^4 w}{\partial x^4} - \left(\mu \frac{EI}{GF_s} + \rho I\right) \frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{\rho I}{GF_s} \frac{\partial^4 w}{\partial t^4} + \\ + \mu \frac{\partial^2 w}{\partial t^2} + c\left(w - \frac{EI}{GF_s} \frac{\partial^2 w}{\partial x^2} + \frac{\rho I}{GF_s} \frac{\partial^2 w}{\partial t^2}\right) = 0 \end{aligned} \tag{4.12}$$

The solution of Eq. (4.12) is

$$w(x,t) = -y_0 \sin \omega \left(\frac{x}{v} - t \right) . \tag{4.13}$$

The velocity v of wave propagation is given by

$$\begin{aligned} \frac{1}{v^2} = \frac{1}{2}\left[\frac{\mu}{GF_s} + \frac{\rho I}{EI} - \frac{1}{\omega^2} \frac{c}{GF_s} \right] \\ \pm \sqrt{\frac{1}{4}\left[\frac{\mu}{GF_s} + \frac{\rho I}{EI} - \frac{1}{\omega^2} \frac{c}{GF_s}\right]^2 - \frac{\mu}{GF_s} \frac{\rho I}{EI} + \frac{1}{\omega^2}\left(\frac{\mu}{EI} + \frac{c}{GF_s} \frac{\rho I}{EI}\right) - \frac{1}{\omega^4} \frac{c}{EI}} . \end{aligned} \tag{4.14}$$

In the special case $\omega = \infty$, it follows that

$$v_1 = v_{SH}$$

and

$$v_2 = v_L$$

In the special case $c = \mu \omega^2$, it follows that

$$v_1 = \infty$$

and

$$v_2 = v_L$$

In the general case $c = \alpha\mu\omega^2$ -- where α is a dimensionless parameter, it follows after a lengthy transformation with $\varphi = F/F_s$ that

$$\frac{1}{v^2} = \frac{1}{2}\frac{\rho}{E}\left[2\varphi(1-\alpha)(1+\nu) + 1\right]\left[1 \pm \sqrt{\left(\frac{2\varphi(1-\alpha)(1+\nu) - 1}{2\varphi(1-\alpha)(1+\nu) + 1}\right)^2 + \varkappa\frac{1-\alpha}{[2\varphi(1-\alpha)(1+\nu) + 1]^2}}\,\right] \tag{4.15}$$

where h is the beam depth and $\varkappa = \dfrac{48E}{\varepsilon^2 h^2 \rho}$. Let us assume that

$\omega = 10^{10}\ \frac{1}{\text{sec}}$, as observed by SMEKAL (10)

$$\left.\begin{array}{l} E = 2.1 \times 10^6 \text{ kg cm}^{-2} \\ \rho = 0.8 \times 10^{-6} \text{ kg cm}^{-4} \text{ sec}^2 \end{array}\right\} \text{steel}$$

$h = 1$ cm (≈ 0.4 inch)(arbitrary)

Then, the factor $\varkappa = \dfrac{1}{\omega^2}\dfrac{48}{h^2}\dfrac{E}{\rho}$ equals 1.26×10^{-6}. It can easily be seen that the velocity of wave propagation is very close to that of the special case $\omega = \infty$.

Let us guess for a moment that the crack propagates after reaching its maximum and constant value with the same velocity as the wave propagates as given by Eq. (4.15), then we conclude that there must be a frequency ω' which is much lower than the frequency ω observed by Smekal. In Fig. 2 the ratio of v/v_L is plotted as a function of the parameter α for three different values of the parameter $\varkappa$. It can be seen that ω' must be lower than $10^{-4}\omega$ or than $10^6\ \frac{1}{\text{sec}}$. However, it can not be proved that such low frequencies exist. Thus, the guess mentioned above is not justified. The difference between crack propagation and wave propagation seems to be caused by the time needed for the delivery of stored elastical energy to the endpoint of the crack and for the transformation of this energy into the energy of the surface tension at the crack, as explained by Smekal (10).

5. The strings parallel to the x-axis cause only a slight increase of the moment of inertia I of the beam in the region where the strings perpendicular to the x-axis are not yet ruptured. The tensile stress in the x-direction must be zero at the end of the strings. Thus, these strings parallel to the x-axis participate in the general state of stress only at abscissae x larger than the abscissa of the instantaneous end of the crack. They are strained by the shear forces acting at the boundary between the beams proper and the network of the strings. It can be concluded that the shear stresses at this boundary are very high in the immediate vicinity of the end of the crack.

6. After these general considerations on the differential equations of the elastic lines of beams, let us go back to the concept of Prandtl as represented in Fig. 1. The beam must be subdivided into two branches, the first one from $x = 0$ to $x = \ell$ (where 2ℓ is the instantaneous length of the crack) and the second one from $x = \ell$ to $x = \infty$.

If the effect of the shear forces on the deformation and if the rotational inertia is neglected, the differential equation for the first branch is simply

$$EI \frac{\partial^4 w_I}{\partial x^4} = p - \mu \frac{\partial^2 w_I}{\partial t^2} \tag{6.1}$$

and for the second branch

$$EI \frac{\partial^4 w_{II}}{\partial x^4} = p - \mu \frac{\partial^2 w_{II}}{\partial t^2} - c(b + w_{II}) \tag{6.2}$$

where p is the external lateral load and $2b$ the length of the strings. If $p - cb$ is abbreviated by p_{II}, Eq. (6.2) can be written in the form

$$EI \frac{\partial^4 w_{II}}{\partial x^4} = p_{II} - \mu \frac{\partial^2 w_{II}}{\partial t^2} - c w_{II}. \tag{6.3}$$

The deflection w is a function of the abscissa x and of the time t. It can be written as

$$w = y(x)\,\varphi(t) \tag{6.4}$$

where y is a function of the abscissa x and φ a function of the time. Substitution of solution (6.4) into Eqs. (6.1) and (6.3) yields

$$EI\,\varphi_I \frac{d^4 y_I}{dx^4} = p - \mu\, y_I \frac{d^2\varphi_I}{dt^2} \tag{6.5}$$

and

$$EI\,\varphi_{II} \frac{d^4 y_{II}}{dx^4} = p_{II} - \mu y_{II} \frac{d^2\varphi_{II}}{dt^2} - c\varphi_{II}\, y_{II}\,. \tag{6.6}$$

Using the abbreviations

$$\left.\begin{aligned}
\frac{p}{EI\varphi_I} &= \delta_I \\
\frac{\mu}{EI}\frac{1}{\varphi_I}\frac{d^2\varphi_I}{dt^2} &= 4\varkappa_I^4 \\
\frac{p_{II}}{EI\varphi_{II}} &= \delta_{II} \\
\frac{\mu}{EI}\frac{1}{\varphi_{II}}\frac{d^2\varphi_{II}}{dt^2} + \frac{c}{EI} &= 4\varkappa_{II}^4
\end{aligned}\right\} \tag{6.7}$$

we obtain

$$\frac{d^4 y_I}{dx^4} = \delta_I - 4\varkappa_I^4\, y_I \tag{6.8}$$

and

$$\frac{d^4 y_{II}}{dx^4} = \delta_{II} - 4\varkappa_{II}^4\, y_{II} \tag{6.9}$$

The solution of Eqs. (6.8) and (6.9) can be taken in the form

$$y_I = \frac{\delta_I}{4\varkappa_I^4} + \sum_1^4 A_{nI}\, e^{\lambda_{In} x} \tag{6.10}$$

and

$$y_{II} = \frac{\delta_{II}}{4\varkappa_{II}^4} + \sum_1^4 A_{nII}\, e^{\lambda_{IIn}x} \tag{6.11}$$

where

$$\lambda_n = \pm\varkappa(1 \pm i) \qquad (i = \sqrt{-1}) \tag{6.12}$$

7. If the influence of the shear stresses does not seem to be negligibly small, the differential equations are

$$EI\,\frac{\partial^4 w_I}{\partial x^4} = p - \mu\,\frac{\partial^2 w_I}{\partial t^2} + \mu\,\frac{EI}{GF_s}\,\frac{\partial^4 w_I}{\partial t^2 \partial x^2} \tag{7.1}$$

and

$$EI\,\frac{\partial^4 w_I}{\partial x^4} = p - c(b + y_{II}) - \mu\,\frac{\partial^2 w_{II}}{\partial t^2} + \mu\,\frac{EI}{GF_s}\,\frac{\partial^2 w_{II}}{\partial t^2 \partial x^2} \tag{7.2}$$

Substituting solution (6.4) again and using the abbreviations

$$\left.\begin{aligned}
\frac{p}{EI\varphi_I} &= \delta_I \\
\frac{\mu}{EI}\,\frac{1}{\varphi_I}\,\frac{d^2\varphi_I}{dt^2} &= 4\varkappa_I^4 \\
\frac{\mu}{GF_s}\,\frac{1}{\varphi_I}\,\frac{d^2\varphi_I}{dt^2} &= 2\psi_I^2 \\
\frac{p - cb}{EI\varphi_{II}} &= \delta_{II} \\
\frac{c}{EI} + \frac{\mu}{EI}\,\frac{1}{\varphi_{II}}\,\frac{d^2\varphi_{II}}{dt^2} &= 4\varkappa_{II}^4 \\
\frac{\mu}{GF_s}\,\frac{1}{\varphi_{II}}\,\frac{d^2\varphi_{II}}{dt^2} &= 2\psi_{II}^2
\end{aligned}\right\} \tag{7.3}$$

we obtain similarly

$$\frac{d^4 y}{dx^4} - 2\psi^2\,\frac{d^2 y}{dx^2} + 4\varkappa^4 y - \delta = 0 \tag{7.4}$$

The solution of this equation can be taken as

$$y = \frac{\delta}{4\varkappa^4} + \sum_1^4 B_n e^{\lambda_n x} \tag{7.5}$$

where the exponents λ_n are given by

$$\lambda_n = \pm\sqrt{\psi^2 \pm \sqrt{\psi^4 - 4\varkappa^4}} \tag{7.6}$$

Usually $\varkappa$ is much larger than ψ; thus, to a first approximation

$$\lambda_n = \pm\varkappa\left[\left(1 + \frac{\psi^2}{4\varkappa^2} + \frac{\psi^4}{32\varkappa^4}\right) \pm i\left(1 - \frac{\psi^2}{4\varkappa^2} + \frac{\psi^4}{32\varkappa^4}\right)\right] \tag{7.7}$$

A comparison of Eqs. (6.12) and (7.7) shows that the numerical values of the exponents λ_n are a little different and thus, the constants of integration A_n must differ from the constants B_n, but nothing is changed in principle.

It is also possible to take the rotational inertia into account if it seems to be desirable. It can be done in the same manner as above.

8. The constants of integration A_{nI} and A_{nII} can be determined such that the boundary conditions at $x = 0$, $x = \ell$ and $x = \infty$ are satisfied. They are as follows:

At $x = 0$, the shear force Q must vanish and the deflection line must have a horizontal tangent.

At $x = \ell$, the deflection y_I must be equal to the deflection y_{II} the slope $\frac{dy_I}{dx}$ must be equal to the slope $\frac{dy_{II}}{dx}$ the second and the third derivative of the deflection of branch I and branch II must be equal to each other whereby the equilibrium of the bending moments at the end of both branches and of the shear forces, respectively, is established.

At $x = \infty$, the bending moment and the shear force must be zero. A system of eight linear equations can be derived from these boundary conditions which can be solved for the eight unknown quantities A_{nI} and A_{nII}. As soon as the constants of integration A_{nI} and A_{nII} are known, the deflection line can be drawn.

9. It can be shown that the inflection point of these bending lines is very close to the abscissa $x = \ell$, i.e., to the end of the crack. This observation suggests a further simplification since as the bending moment is zero at the inflection point. This means that the shear force Q at the abscissa $x = \ell$ can be determined from the condition of equilibrium in the y-direction alone. It follows that

$$Q = \ell p - \frac{d(mv_y)}{dt} \tag{9.1}$$

where m is the mass (variable with time $= \mu\ell$) and

v_y is the velocity with which the point $x = \ell$ travels in the y-direction or $v_y = \frac{\partial y}{\partial t}$.

The velocity of propagation of the crack will be denoted by $v_x = \frac{d\ell}{dt}$. In order to find the relation between v_y and v_x, let us consider a point at the distance dx from the end of the crack. If the stress in the string at $x = \ell$ is σ_0, then the stress in the string at $x = \ell + dx$ is

$$\sigma = \sigma_0 - \frac{\partial \sigma}{\partial x} dx. \tag{9.2}$$

The crack travels the distance dx in the time dt. At the time $t + dt$, the stress at the point $x = \ell + dx$ will again equal σ_0. Thus, the increase $\frac{\partial \sigma}{\partial x} dx$ must occur during the time dt. Since $\sigma = c(b + y)$, we have

$$\frac{\partial \sigma}{\partial x} dx = \frac{c\, \partial(b+y)}{\partial x} dx = c \frac{\partial y}{\partial x} d\ell \tag{9.3}$$

on the other hand

$$\frac{\partial\sigma}{\partial t}\,dt = \frac{c\,\partial(b+y)}{\partial t}\,dt = c\,\frac{\partial y}{\partial t}\,dt \tag{9.4}$$

Thus

$$-c\,\frac{\partial y}{\partial x}\,d\ell = c\,\frac{\partial y}{\partial t}\,dt$$

or

$$-v_y = v_x\left(\frac{\partial y}{\partial x}\right) \tag{9.5}$$

This relation has already been derived by L. Prandtl (5).

The following calculations will be simpler if the origin of the system of coordinates is assumed to coincide with the endpoint of the crack. The subscript II may also be omitted. According to Eq. (6.11) the bending line is given by

$$y = \frac{\delta}{4\varkappa^4} + \sum_1^4 A_n\, e^{\lambda_n x}$$

or in order to avoid the complex functions

$$y = \frac{\delta}{4\varkappa^4} + D_1 e^{-\varkappa x}\cos\varkappa x + D_2 e^{-\varkappa x}\sin\varkappa x + D_3 e^{+\varkappa x}\cos\varkappa x + D_4 e^{+\varkappa x}\sin\varkappa x \tag{9.6}$$

Using the boundary conditions at $x = \infty$ we see that D_3 and D_4 must be zero. The bending moment at $x = 0$ is assumed to be zero as pointed out at the beginning of this section. This condition is satisfied by taking

$$D_2 = 0 \tag{9.7}$$

The fourth boundary condition is

$$Q = EI\left(\frac{d^3y}{dx^3}\right)_{x=0} \tag{9.8}$$

and is satisfied if

$$D_1 = \frac{Q}{2EI\varkappa^3} \tag{9.9}$$

Thus, we have

$$y = \frac{\delta}{4\varkappa^4} + \frac{Q}{2EI\varkappa^3} e^{-\varkappa x} \cos \varkappa x \tag{9.10}$$

It must be mentioned that it has been **tacitly** assumed that the term $\frac{1}{\varphi}\frac{d^2\varphi}{dt^2}$ in the abbreviation $\varkappa$ - sec Eq. (6.7) is a constant.

The slope of the elastic line at $x = 0$ is

$$\frac{dy}{dx} = -\frac{Q}{2EI\varkappa^2} \tag{9.11}$$

Substituting Eq. (9.11) into Eq. (9.5) yields

$$v_y = \frac{Q}{2EI\varkappa^2} v_x \quad . \tag{9.12}$$

Substituting Eq. (9.12) into Eq. (9.1) yields

$$Q = \ell p - \frac{1}{2EI\varkappa^2} \frac{d(mQv_x)}{dt} \tag{9.13}$$

or

$$Q = \ell p - \frac{\mu}{2EI\varkappa^2} \ell \frac{d(Qv_x)}{dt} - \frac{\mu}{2EI\varkappa^2} Qv_x \frac{d\ell}{dt} \; . \tag{9.14}$$

At the abscissa $x = 0$, the deflection y_0 is

$$y_0 = \frac{\delta}{4\varkappa^4} + \frac{Q}{2EI\varkappa^3} \; . \tag{9.15}$$

The tensile stress σ_0 at the abscissa $x = 0$ is

$$\sigma_0 = c(b+y_0)$$

or

$$\sigma_0 = cb + \frac{c\delta}{4\varkappa^4} + \frac{cQ}{2EI\varkappa^3} \; . \tag{9.16}$$

Solving Eq. (9.16) for Q yields

$$Q = \frac{2EI\varkappa^3}{c} \left(\sigma_0 - cb - \frac{c\,\delta}{4\varkappa^4}\right) . \tag{9.17}$$

Since σ_0 is the maximum stress at which the rupture occurs, it is independent of the time t; thus $\frac{d\sigma_0}{dt} = 0$. Consequently

$$\frac{dQ}{dt} = 0. \tag{9.18}$$

Differentiating Eq. (9.14) with respect to time we have

$$\frac{dQ}{dt} = p\,\frac{d\ell}{dt} - \frac{\mu}{2EI\varkappa^2}\,\frac{d}{dt}\left[\ell\,\frac{d(Qv_x)}{dt}\right] - \frac{\mu}{2EI\varkappa^2}\,\frac{d}{dt}\left[Qv_x\,\frac{d\ell}{dt}\right] . \tag{9.19}$$

or with $v_x = \frac{d\ell}{dt}$ and the abbreviation

$$q = \frac{p}{\mu\varkappa\left(\frac{\sigma_0}{c} - b - \frac{\delta}{4\varkappa^4}\right)} \tag{9.20}$$

and using Eq. (9.18) we obtain

$$\ell\,\frac{d^3\ell}{dt^3} + 3\,\frac{d\ell}{dt}\,\frac{d^2\ell}{dt^2} - q\,\frac{d\ell}{dt} = 0 \tag{9.21}$$

This equation does not contain the time t explicitly. Therefore, we consider t as a function of ℓ and we set

$$d\ell/dt = f(\ell). \tag{9.22}$$

Then

$$\frac{d^2\ell}{dt^2} = \frac{df}{dt} = \frac{df}{d\ell}\,\frac{d\ell}{dt} = f\,\frac{df}{d\ell} \tag{9.23}$$

and

$$\frac{d^3 \ell}{dt^3} = f^2 \frac{d^2 f}{d\ell^2} + f \left(\frac{df}{d\ell}\right)^2 \tag{9.24}$$

Thus Eq. (9.21) becomes

$$\ell \left[f^2 \frac{d^2 f}{d\ell^2} + f \left(\frac{df}{d\ell}\right)^2 \right] + 3f^2 \frac{df}{d\ell} - qf = 0 \tag{9.25}$$

If $f \neq 0$,

$$\ell f \frac{d^2 f}{d\ell^2} + \ell \left(\frac{df}{d\ell}\right)^2 + 3f \frac{df}{d\ell} - q = 0 \tag{9.26}$$

Next, we set

$$f = \sqrt{g(\ell)} \tag{9.27}$$

Consequently

$$df = \frac{1}{2} \frac{dg}{\sqrt{g}} \tag{9.28}$$

and

$$d^2 f = -\frac{1}{4} \frac{dg^2}{g\sqrt{g}} + \frac{1}{2} \frac{d^2 g}{\sqrt{g}} \tag{9.29}$$

Substituting the Eqs. (9.27), (9.28) and (9.29) into Eq. (9.26) yields

$$\frac{d^2 g}{d\ell^2} + \frac{3}{\ell} \frac{dg}{d\ell} - \frac{2q}{\ell} = 0 \tag{9.30}$$

This linear equation has the general solution

$$g(\ell) = c_0 + c_1 \frac{1}{\ell^2} + \frac{2}{3} q\ell \tag{9.31}$$

Thus, the velocity of propagation of the crack is given by

$$v_x = \frac{d\ell}{dt} = \sqrt{c_0 + c_1 \frac{1}{\ell^2} + \frac{2}{3} q\ell} \tag{9.32}$$

If the velocity v_{x_0} and the acceleration $\left(\frac{dv_x}{dt}\right)_0$ are known at a time t when the crack has a length ℓ_0, the constants of integration c_0 and c_1 can be determined. It follows that

$$
\begin{aligned}
c_1 &= -\ell_0^3\left[\frac{1}{2}\left(\frac{dv_x}{dt}\right)_0 - \frac{1}{3}\ q\right] \\
c_0 &= v_{x_0}^2 + \frac{\ell_0}{2}\left(\frac{dv_x}{dt}\right)_0 - q\,\ell_0 .
\end{aligned}
\tag{9.33}
$$

Thus

$$
v_x = \sqrt{\ell_0}\sqrt{\left[\frac{v_{x_0}^2}{\ell_0} + \frac{1}{2}\left(\frac{dv_x}{dt}\right)_0 - q\right] - \left[\frac{1}{2}\left(\frac{dv_x}{dt}\right)_0 - \frac{1}{3}\,q\right]\left(\frac{\ell_0}{\ell}\right)^2 + \frac{2}{3}\,q\left(\frac{\ell}{\ell_0}\right)} \tag{9.34}
$$

10. In the paper of Smekal (10), an experimental curve is published, which gives the velocity v_x of the propagation of the crack as a function of the length of the crack. The experiments have been made with cylindrical rods of glass. In spite of the fact that our considerations concern a plate, we tried to approximate the experimental curve by an equation of the type (9.32). We found

$$
\frac{v_x}{v_{SH}} = \sqrt{0.1482 - 0.04033\left(\frac{L}{\ell}\right)^2 + 0.4389\,\frac{\ell}{L}} \tag{10.1}
$$

where ℓ_0 of the Eq. (9.34) is replaced by the unit of length L which we selected to be 0.1 cm. Fig. 3 shows how close the curve representing Eq. (10.1) comes to the test results. We certainly do not believe that this agreement can be considered as a proof of our considerations but it shows that this attempt is not too far from the truth.

The main objection against the comparison above is the fact that a the area of the crack is assumed to be proportional to the length of the crack while Smekal's experiments on the cylindrical rods show that the border line of the crack is a circle, see Fig. 4. In the domain b for which the calculations of section 8 are made, the area of the crack is not a linear function of the length of the crack.

11. If it is desired to know the length of the crack as a function of the time, the integral

$$t - t_1 = \int_{\ell_1}^{\ell} \frac{\ell \, d\ell}{\sqrt{c_1 + c_0 \ell^2 + \frac{2}{3} q \ell^3}} \tag{11.1}$$

-- see Eqs. (9.32), (9.33) and (9.34) -- must be solved where the subscript 1 indicates the beginning of the rapid increase of the crack propagation. This is an elliptic integral.

If numerical values of c_1, c_0 and q are known as, for instance, in Eq. (10.1), it is much easier to employ a mechanical summation. The curve shown in Fig. 5 has been calculated in this way. The subscript 2 indicates the beginning of the constant velocity of the crack propagation. The time $t_2 - t_1$ equals $\frac{1.595L}{v_{SH}}$.

BIBLIOGRAPHY

(1) A. A. Griffith: "The Phenomena of Rupture and Flow in Solids", Phil. Trans. Roy. Soc. London, A221 (1920) p. 163.

(2) A. A. Griffith: "The Theory of Rupture", Proceedings of the First International Congress of Applied Mechanics, Delft (1924) p. 55.

(3) A. Smekal: "Technische Festigkeit und molekulare Festigkeit", Die Naturwissenschaften, vol. 10 (1922) p. 799.

(4) K. Wolf: "Zur Bruchtheorie von A. A. Griffith", Zeitschrift fuer Angewandte Mathematik und Mechanik, vol. 3 (1923) p. 107.

(5) L. Prandtl: "Ein Gedanken modell fuer den Zerreissvorgang sproeder Koerper", Zeitschrift Fuer Angewandte Mathematik und Mechanik, vol. 13, (1933) p. 129.

(6) E. L. Yoffe: "The Moving Griffith Crack", The philosophical Magazine, vol. XLII -- seventh series, London (1951) pp. 739-750.

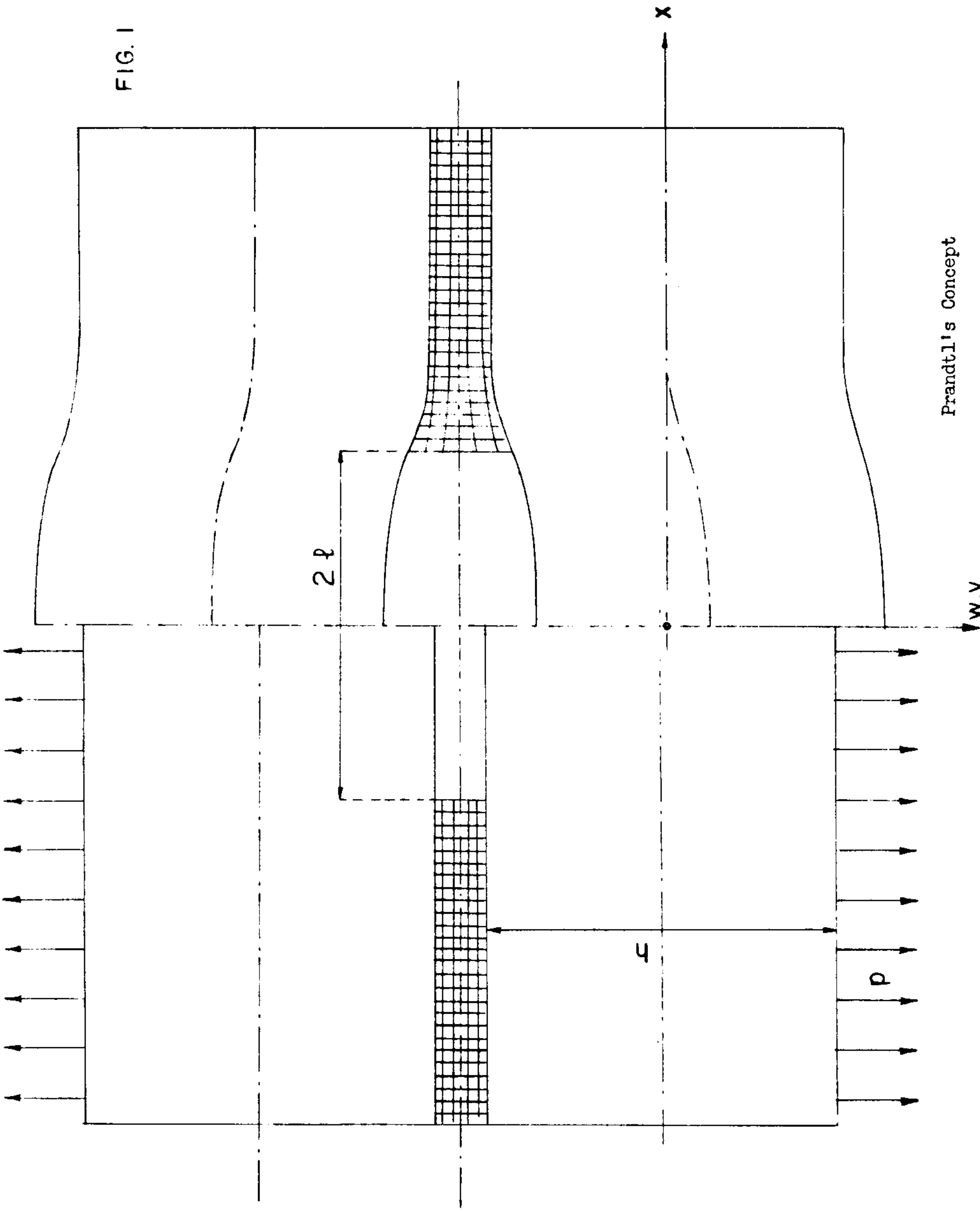
(7) H. R. H. Schardin & W. Struth: "Zeitschrift fuer Technische Physik", vol. 18 (1937) p. 747.

(8) A. Smekal: "Dynamik des sproeden Zugbruches von zylindrischen Glasstaeben", Acta Physica Austriaca, vol. 7 (1953) p. 110.

(9) A. Smekal: "Zum Bruchvorgang bei sproeden Stoffverhalten unter ein-und mehrachsigen Beanspruchungen", Oesterreichisches Ingenieur Archiv, vol. 7 (1953) p. 49.

(10) A. Smekal: "Zur Physik des sproeden Stoffverhaltens, 'Radex-Rundschau'", (Oesterreichisch-Amerikanische Magnesit Aktiengesellschaft, Radentheim-Kaernten) (1953) p. 4.

FIG. 1



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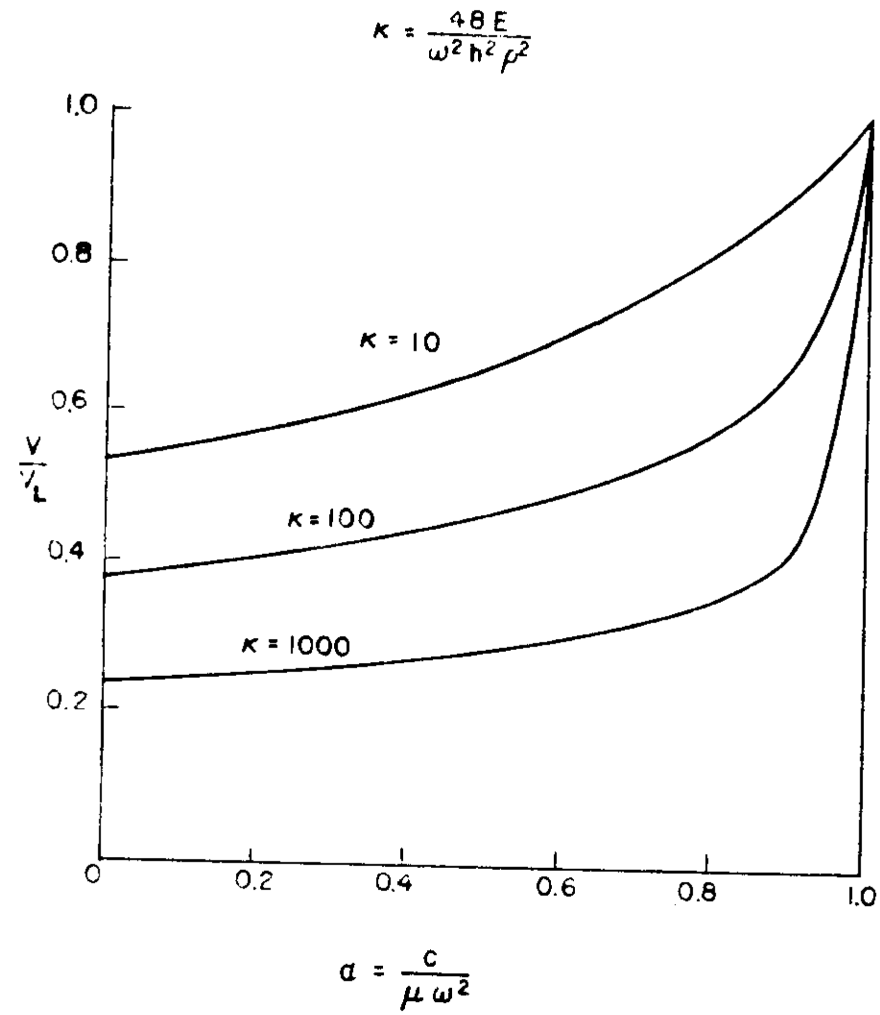

FIG. 2

v/v_L as function of α and $\varkappa$

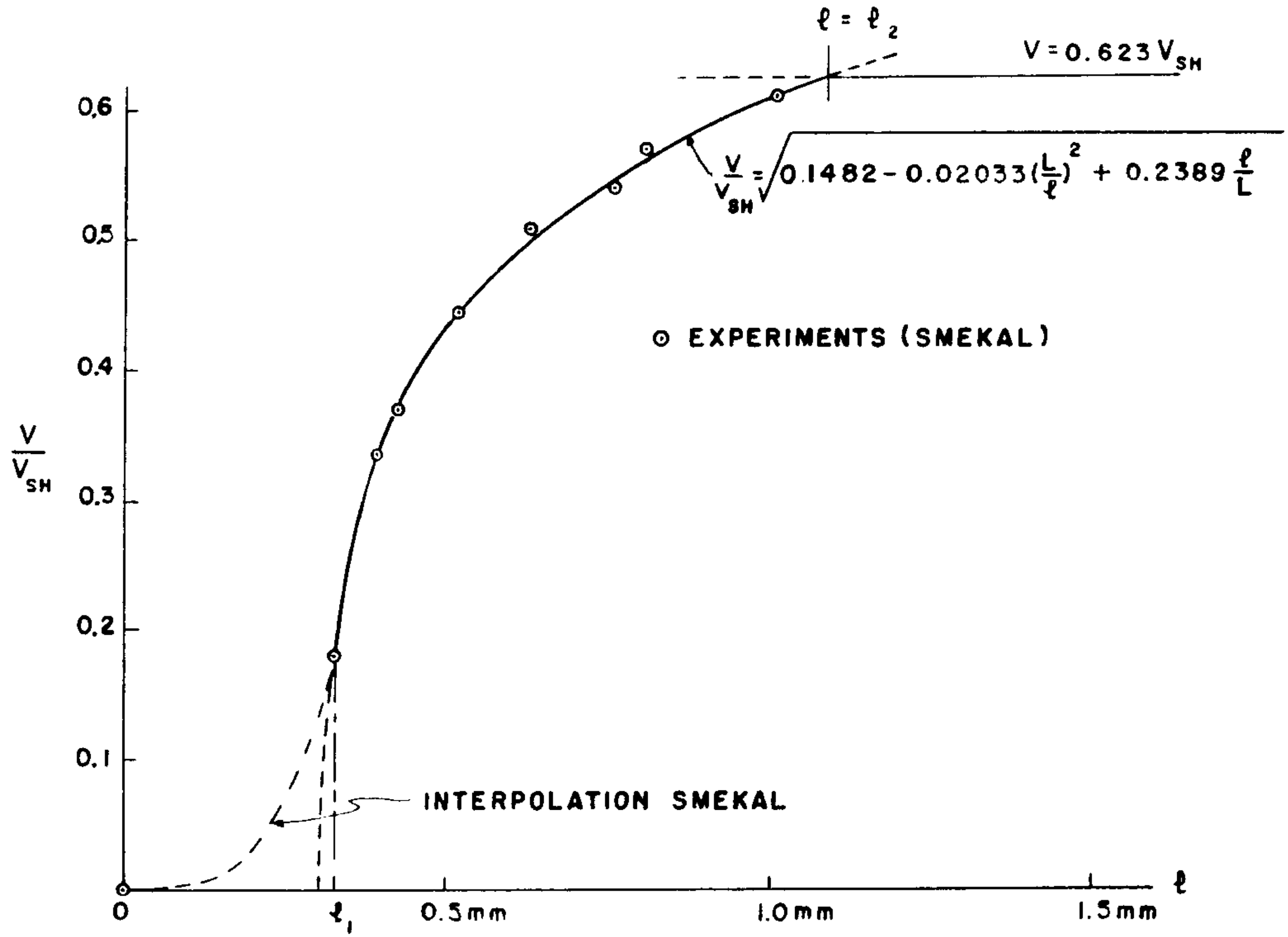

FIG. 3

v_{crack}/v_{SH} vs. length of the crack

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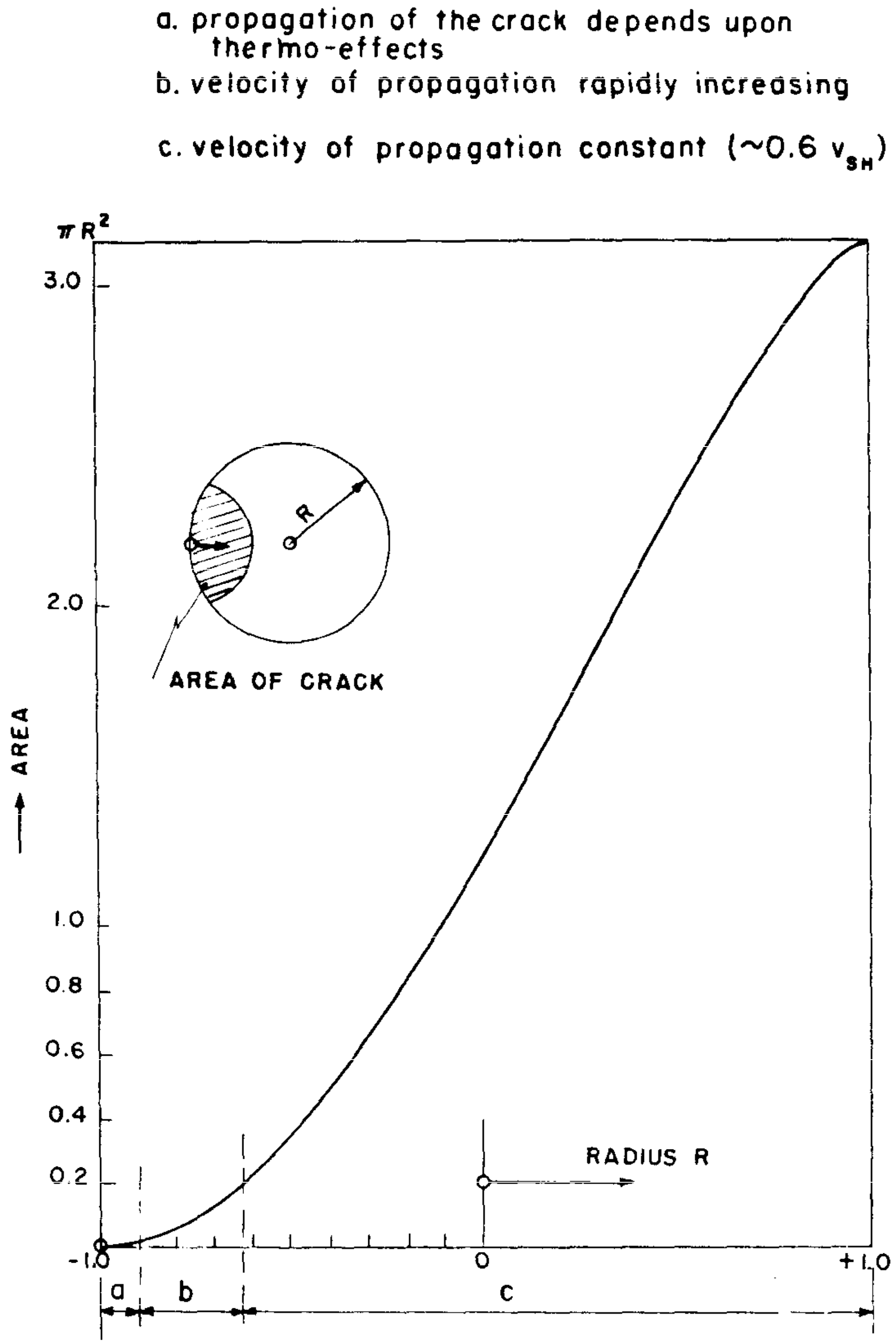

FIG. 4

A_{crack} vs. length of the crack

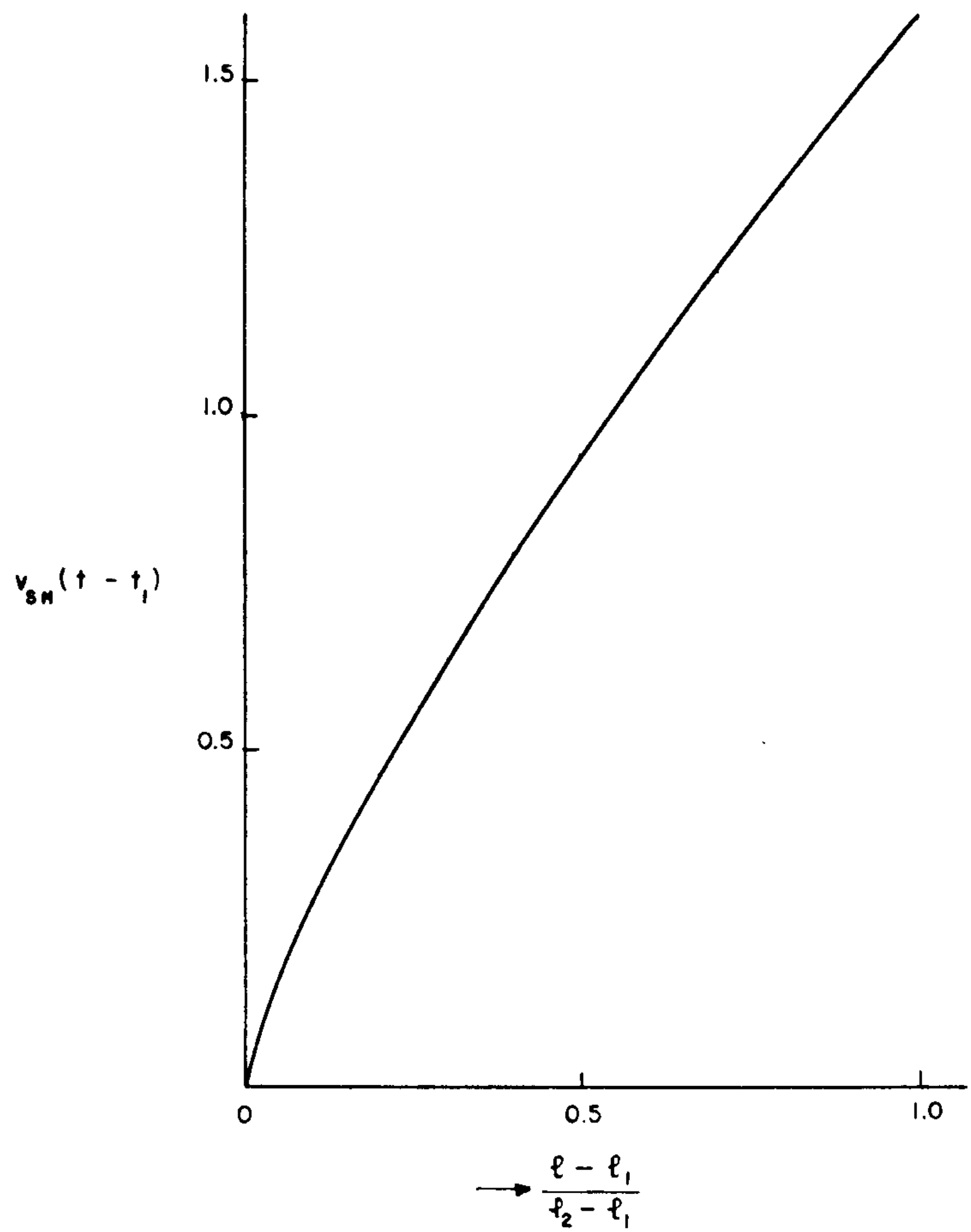

FIG. 5

Time vs. length