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On the Determination of Design Wave Height

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ABSTRACT

A study of the largest value distribution for a zero-mean Gaussian process is presented. A first-passage problem is implemented to derive the largest value distribution and the associated limiting decay rate is newly proposed assuming that the process is narrow-banded and the upcrossings of process are correlated. A consideration is also given to the non-stationary and the long-term prediction of the largest value. For a typical wave data, the design wave heights are determined using the largest value distribution.

INTRODUCTION

Naval Architects and Ocean Engineers have aspired to provide a rational procedures by which the field industry and regulatory agencies can determine the most "adequate" design loads. The structures must withstand the design loads safely and at the same time economically during their service lifetimes. The task of assessing the most adequate and consistent safety margin can be done most effectively using the reliability theory which requires load distributions and resistance (or capacity) distributions. To comply with this aspect, during the past two decades, load distributions have become a central interest with the ultimate objective of improving the prediction of wave loads and the rational design procedures.

In connection with the load distribution, as a tool for determining the design wave height, the largest value distribution of a random variable which can be described by a zero-mean Gaussian process, is presented by this paper. A first-passage problem is implemented to derive the largest value distribution and the associated limiting decay rate is newly proposed assuming that the process is narrow-banded and the upcrossings of the process are correlated. Some methodological extension is given to the non-stationary and the long-term prediction of the largest value of a random variable.

For a typical long-term wave scatter data, the probability of exceedance and the probability density function of the largest value of wave heights are computed for the selected design periods, 1, 10, 25, 50, 100 years. Their numerical values are discussed in comparison with those obtained by the conventional method, that is, the Weibull distribution fit and the peak analysis. The largest value distribution method clearly quantifies the design wave height as a random variable in a probabilistic manner and can serve when implementing the reliability theory which requires the load distribution and the resistance distribution. Several advantages of the largest value distribution method, in fact, based on upcrossing analysis are discussed as a substitution for the conventional method based on peak analysis in determining the design wave height.

THEORY

The Largest Value Distribution (LVD)

The first-passage probability $p(b;t|x_0)$ is defined such that the value of the process $X(t)$ surpasses a certain fixed threshold b for the first time during t to $t+dt$ on the interval $[0, T]$, starting from probabilistic initial condition $X_0 = x_0$. This is intimately related with the probability $S(b;T|x_0)$ that the largest value X of the process during the interval T remains smaller than the threshold b, i.e., in a mathematical form:

$$S(b;T|x_0) = 1 - \int_0^T p(b;t|x_0)dt \qquad (1)$$

where $S(b;T|x_0) = \text{prob}\{X_L < b \ ; \ 0<t<T\}$; i.e., the probability of no-excursion (say, the fraction of the ensembles which do not have an excursion). In other words, it is exactly the cumulative distribution function (CDF) of the largest value X_L of a random process $X(t)$ during the time interval T.

When the threshold b is associated with the failure surface of the reliability theory, $S(b;T|x_0)$ represents the probability that the system has not failed at time T, starting from the initial condition $X_0 = x_0$ in safe domain and is called the reliability of the system.

For a diffusion Gaussian-Markov process, the probability of no-excursion, $S(b;T|x_0)$, that is, the cumulative distribution function of the largest value during the interval T is the solution for the backward Kolmogorov differential equation [12,15]

$$\frac{\partial S}{\partial T} = f\frac{\partial S}{\partial x_0} + \frac{1}{2}\sigma^2\frac{\partial^2 S}{\partial x_0^2} \quad (2)$$

in which f is a drift and σ is a diffusion coefficient, with the boundary and initial conditions:

$$S(b;T|b) = 0 \quad (3)$$

$$S(b;T|0) = \text{finite} \quad (4)$$

$$S(b;0|x_0) = 1, \; 0 < x_0 < b \quad (5)$$

The formulation of the backward Kolmogorov differential equation with the associated initial and boundary conditions can be studied in detail by referring to [8].

Approximate Solution and Limiting Decay Rate

For the backward Kolmogorov equation with initial and boundary conditions given in the equations (2) through (5), unfortunately, an exact solution for $S(b;T|x_0)$ has not yet been obtained in an analytical form even for a simple single-degree-of-freedom linear oscillator subject to a white noise excitation. The best approximate solution can be given, by separation of variables, in the form of eigenfunction expansion as presented in reference [7]; that is,

$$S(b;T|x_0) = \sum_{i=0} A_i \exp(-\alpha_i T) \quad (6)$$

The values of A_i and α_i depend in a complicated manner on the characteristics of functions involved in the equation (2), on the boundary and initial conditions imposed as well as on the magnitude b of the threshold. However, Mark[9] found in his simulation studies that for the larger but still stationary period T, which is of most practical interest, in comparison with the mean period defined by the zero upcrossings of the process, the solution $S(b;T|x_0)$ tends to $S(b;T)$, i.e., becomes independent of the initial condition. He also found that for high threshold b, say $b \geq 2\sqrt{m_0}$, where $\sqrt{m_0}$ is the root mean square of the process $X(t)$ on [0,T], the coefficients $\alpha_0 \longrightarrow 0$ but small and $A_0 \longrightarrow 1$, while $A_i \longrightarrow 0$ for every $i \geq 1$. This leads to the drastic simplification of the approximate solution with good accuracy.

For high thershold b and large period T, the approximate solution in the limiting form can be given by

$$S(b;T) \cong \exp(-\alpha T) \quad (7)$$

where α is the smallest of α_i in the equation (6). The value of α is the *dominant eigenvalue* of the equation (6) and is called the limiting decay rate. The larger value of α implies that the process $X(t)$ is likely to excurse beyond the specified bound in a period. On the other hand, smaller value of α implies that the process $X(t)$ is unlikely to excurse beyond the specified bound in the same period.

Consequently, the problem of solving the backward Kolmogorov equation with initial and boundary conditions for the no-excursion probability distribution function $S(b;T)$ that the largest value of the process $X(t)$ during the time interval T remains smaller than the threshold b, has been reduced to that of determining the value of the limiting decay rate α. That is, once we find the limiting decay rate α, we can obtain the largest value distribution of the process $X(t)$ during the time interval T.

Limiting Decay Rate Based on Correlated Peaks

Concerning with the limiting decay rates, many versions of estimates have been proposed. These include the assumption of independent crossings, that of independent maxima (or peaks), that of independent envelope crossings, that of independent envelope peaks and that of two-state Morkov process, etc. [5]

The estimation of α using the envelope should be confined to the narrow band process since the mathematical definition of the envelope by Rice [11] or Crandall [4] does not represent well the envelope of the wide band process. On the other hand, for the narrow band process, the assumption of independent crossings or independent maxima appears not to be appropriate since the grouping tendency (clumping) can be significantly noticed.

Vanmarcke(1975) improved the limiting decay rate based on the envelope, taking into account the fraction of time that the envelope spends above the threshold b and the fact that not all the envelope crossings (E-crossings) will be followed by a crossing of the process itself (B-crossing) during the time interval that the envelope stays above the threshold. He evaluated the fraction ϕ of the so-called "qualified" E-crossings which are followed immediately by at least one B-crossing [13], which is of importance for high threshold level and is of form

$$\phi = 1-\exp(-\sqrt{2\pi}\, q\eta) / (\sqrt{2\pi}\, q\eta) \quad (8)$$

in which $\eta = b/\sqrt{m_0}$: non-dimensional threshold

$q = \sqrt{1 - m_1^2 / m_0 m_2}$

m_i = i-th moment of the power spectral density (PSD) function of the process $X(t)$

In considering the fraction of time that the envelope spends above the threshold, which is of importance for the lower threshold level, he modified the two-state process for the envelope by transferring from state 1 (above the threshold) to state 0 (below the threshold) those intervals T_1's in which a B-crossing does not occur. Further, assuming that the expected time intervals $E[T_0']$ and $E[T_1']$ in the modified process are increased in the same ratio ϕ , Vanmarcke finally proposed the improved limiting decay rate such as

$$\alpha = N^+(\eta) \cdot \frac{1-\exp(-\sqrt{2\pi}q\eta)}{1-\exp(-\eta^2/2)} \quad (9)$$

in which $N^+(\eta)=(1/2\pi)\sqrt{m_2/m_0}\exp(-\eta^2/2)$: the expected rate of B-crossings.

The envelope is assumed to pass randomly back and forth from the state 0 to the state 1. The successive intervals T_0 and T_1 spent in states 0 and 1 are taken to be independent random variables and exponentially distributed. The up-crossings of the envelope across the threshold η is taken to be "recurrent" events, i.e., events whose recurrent time intervals are independent and identically distributed. This constitutes a two-state Markov process and for successive pairs of time intervals,

$$E[T_0+T_1] = 1 / N_R^+(\eta) \quad (10)$$

where $N_R^+(\eta)= \sqrt{2\pi}\, q\eta\, N^+(\eta)$:the expected rate of E-crossings.

For a true narrow band process, since the envelope has a Rayleigh distribution, the fraction of time that the envelope stays above the threshold η can be obtained by integrating the Rayleigh probability density function from η to infinity; therefore,

$$E[T_1]/E[T_0+T_1] = \int_\eta^\infty \eta\exp(-\eta^2/2)d\eta = N^+(\eta)/N^+(0) = \exp(-\eta^2/2) \quad (11)$$

where $N^+(\eta)$ is the expected rate of B-crossings and $N^+(0)$ is the expected zero up-crossing rate.

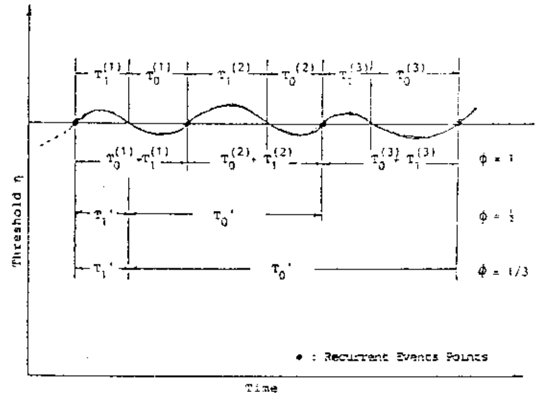

Fig.1 Visualization of times T_0 and T_1

The time intervals of T_0, T_1 and T_0+T_1 are sketched in Fig. 1.

In Fig. 1, if $\phi=1$, all the E-crossings are followed immediately by at least a B-crossing. However, if $\phi=1/2$, only a half of total E-crossings are qualified. To consider this, in Fig. 1, if a B-crossing does not occur during the second time interval $T_1^{(2)}$ spent above the threshold η , then we will modify the two-state process by transferring the second time interval $T_1^{(2)}$ into the modified time interval T_0'. Consequently, the time that the envelope spends above the threshold η is $T_1^{(1)}$ again, i.e., $T_1'=T_1^{(1)}$ and the time that the envelope spends below the threshold becomes

$$T_0' = T_0^{(1)} + T_0^{(2)} + T_1^{(2)} \quad (12)$$

If $\phi = 1/3$, i.e., when a B-crossing does not occur during the second and the third time intervals, $T_1^{(2)}$ and $T_1^{(3)}$ in which the envelope spends above the threshold η , then the modified time interval of state 0 will be

$$T_0' = T_0^{(1)} + T_0^{(2)} + T_0^{(3)} + T_1^{(2)} + T_1^{(3)} \quad (13)$$

while the time interval above the threshold remains unchanged, that is, $T_1' = T_1$. It is important to note that in a modified two-state process, the time that the envelope spends above the threshold does not change regardless of the fraction ϕ of "qualified" E-crossings. This is different from the Vanmarcke's assumption that $E[T_0']$ and $E[T_1']$ in the modified process are increased in the same ratio ϕ ; that is, $E[T_0'] = \phi\, E[T_0]$ and $E[T_1'] = \phi\, E[T_1]$.

As a result, the expected time interval of recurrent events will be

$$E[T_0'+T_1'] = 1/ \phi N_R^+(\eta) \quad (14)$$

and $E[T_1']$ is assumed unchanged from $E[T_1]$, that is,

$$E[T_1'] = E[T_1] = \frac{1}{N_R^+(\eta)} \cdot \frac{N^+(\eta)}{N^+(0)} \quad (15)$$

From the equations (14) and (15), the expected time $E[T_0']$ that the envelope spends below the threshold η in a modified two-state process can be given by

$$E[T_0'] = \frac{1}{\phi N_R^+(\eta)} - \frac{1}{N_R^+(\eta)} \frac{N^+(\eta)}{N^+(0)} \qquad (16)$$

Assuming that the first passage time T_0' in a modified two- state process is exponentially distributed, we will have

$$\text{prob}\{T_0' < t\} = 1 - \exp(-\alpha t) \qquad (17)$$

and $E[T_0'] = 1/\alpha$.

Therefore, the no-excursion probability is

$$S(\eta;T) = \text{prob}\{T_0' > T\} = \exp(-\alpha T) \qquad (18)$$

and the limiting decay rate α becomes

$$\alpha = 1/E[T_0'] = \phi N_R^+(\eta) / \left\{1 - \phi \frac{N^+(\eta)}{N^+(0)}\right\}$$
$$= N^+(\eta) \frac{1 - \exp(-\sqrt{2\pi} q \eta)}{1 - \phi \exp(-\eta^2/2)} \qquad (19)$$

Comparing the equation (19) with the equation (9), we can notice that the parameter ϕ is again the fraction of "qualified" E-crossings and the new limiting decay rate introduces the fraction ϕ as a correction factor to Vanmarcke's limiting decay rate. New limiting rate is hoped to correct Vanmarcke's which has been found to systematically overpredict the simulation result [10].

New limiting decay rate in the equation (19) is plotted for various q values in Fig. 2 and, for q= 0.1, 0.5, 1.0, is compared with Vanmarcke's in Fig. 3 through Fig. 5. New limiting decay rate predicts less conservative than Vanmarcke's. It deviates farther from Vanmarcke's and converges faster to $N^+(\eta)$ as q goes to unity.

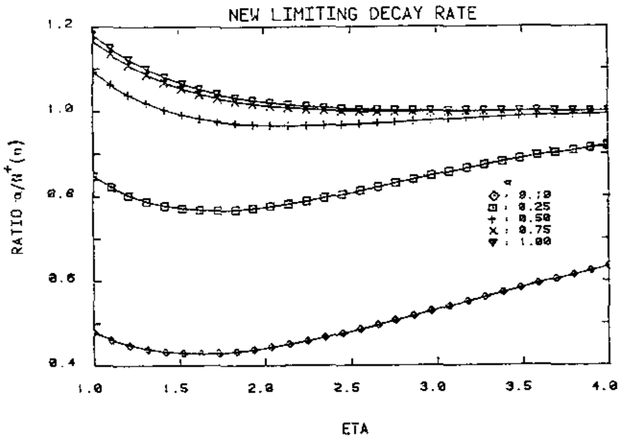

Fig. 2 New limiting decay rates at five different q values

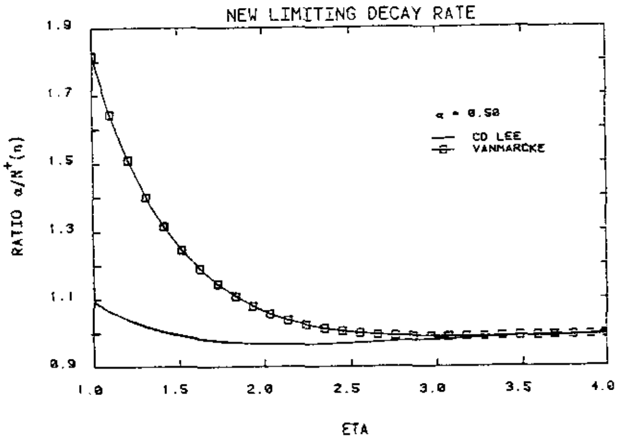

Fig.4 Comparison of new limiting decay rate and Vanmarcke's at q = 0.5

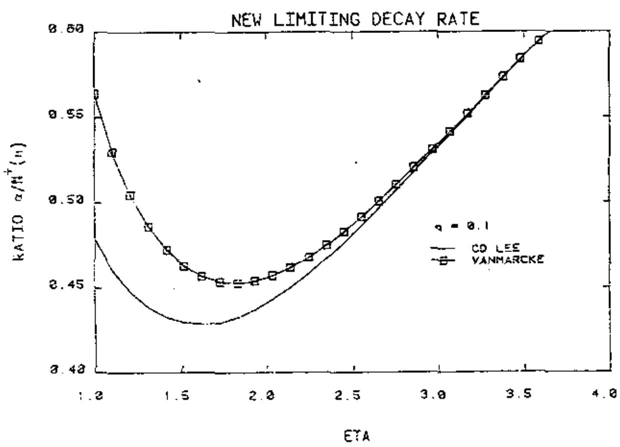

Fig.3 Comparison of new limiting decay rate and Vanmarcke's at q = 0.1

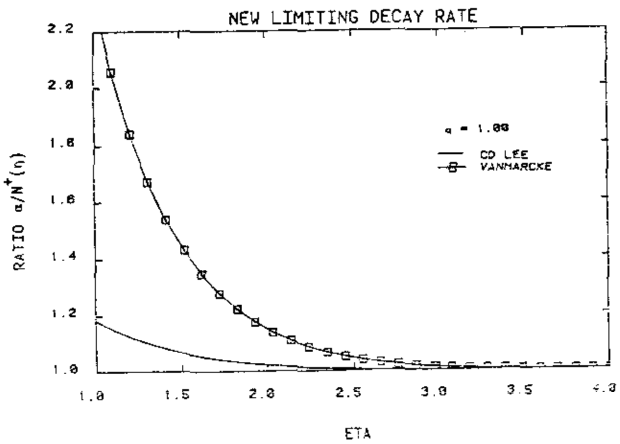

Fig.5 Comparison of new limiting decay rate and Vanmarcke's at q = 1.0

Non-stationary and Long-term Prediction of the Largest Value

In the preceding sections, only a stationary Gaussian process has been considered in determining the limiting decay rates and predicting the largest value of a random process in a stationary period.

However, the limiting decay rates that depend on parameters m_0, $N^+(0)$ and q can be heuristically extended to a non-stationary (time-dependent) process, provided that we are given a time-dependent power spectral density (PSD) function instead of a time-independent PSD function for a stationary process.

With time-dependent limiting decay rate $\alpha(\eta,t)$ which can be defined in terms of the first few time-dependent moments of the time-dependent PSD function, the CDF of the largest value for a non-stationary period T can be given by

$$S(\eta;T)= \exp\{- \int_0^T \alpha(\eta,t)dt\} \qquad (20)$$

For practical design purpose, the long-term prediction of the largest value is important. Since the long-term period should be considered as being non-stationary, a similar argument can be given. However, it seems impractical to find a time-dependent PSD function of a random process throughout a long period T and to directly evaluate the integral given by the equation (20). To overcome this, we may assume that a non-stationary long-term period can be subdivided into multiple stationary short-term periods. Then, the CDF of the largest value becomes

$$S(\eta;T)=\exp[-\sum_i \alpha_i(\eta)\ \Delta t_i n_i] \qquad (21)$$

where n_i; number of occurrences of the i-th state during the period T

Δt_i; time duration in seconds of the stationarity for the i-th state.

$\alpha_i(\eta)$; limiting decay rate for the i-th state.

Equation (21) implies that the largest value of each stationary period is mutually independent and this is well represented by the property of exponential function.

THE LVD OF LONG-TERM WAVE HEIGHT

Stochastic Model for Random Wave

In applying the largest value distribution theory presented herein, ocean wave elevation is assumed to be the sum of many sinusoidal wave components with arbitrary amplitudes, frequencies and uniformly distributed phase angles in $[0, 2\pi]$. This assumption leads to a Gaussian process for the wave elevation by the "Central Limit Theorem" which states that the probability distribution of the sum of statistically independent random variables tends to become Gaussian as the number of independent random variables increases with limit, regardless of the probability distribution of the random variables as long as it has a finite mean and finite variance.

For a stationary short-term period, it is assumed that the random process of wave elevation is successfully described by I.T.T.C two-parameter spectrum; i.e,

$$S(\omega)=(A/\omega^5)\exp(-B/\omega^4) \qquad (22)$$

where $A = 173H_{1/3}^2\ /T^4$

$B = 691/T^4$

ω = circular frequency in rad/sec

$H_{1/3}$ = significant wave height in meters

$T = 2\pi\cdot m_0/m_1$ =characteristic period

m_i = the i-th moment of $S(\omega)$

The use of I.T.T.C. two-parameter spectrum also implies that the process is narrow-banded for a stationary short-term period and their peaks are correlated.

Typical Wave Statistics

As a typical long-term wave data, the synthesized height and period contingency table is obtaind from the British Maritime Technology (BMT) and is given in Fig. 6. It covers 35 years' data and every cell of the contingency provides the number of occurrences in particles per thousand (PPT) of pairs of significant wave height and mean zero-crossing period which characterize the sea states. In addition, the BMT provides the extreme significant wave height for 1, 10, 25, 50, 100 design years, respectively, which describes the extreme sea state, based on the conventional Weibull distribution fit method. It also gives the most probable value of the largest wave height for the extreme sea state based on order-statistics. In their actual computations, the BMT set 12-hous torm period in determining the total number of storms in N years and took the mean wave period T = 15 seconds in calculating the number of wavesin 12 hours. It is noticed that T =5 secs is the specific value for the North Sea and North Atlantic and any justification of its use for other area of interest is not given. More details can be referred to the BMT report [1,2] and these results will be compared later with those of the largest value distribution method.

Numerical Calculations and Discussions

For the BMT ($H_{1/3}$, T) contingency table of a typical wave data, the probability of exceedance and the PDF of the largest value of wave height are calculated for 1, 10, 25, 50, 100 design years. The probability of exceedance is just complementary to the CDF of the largest value and the PDF is the first derivative of the CDF.

The overall calculation procedure is briefly summarized in Fig. 7.

In calculating the moments of I.T.T.C spectrum in the equation (22), the visual wave height H_v in the BMT table is interpreted as the significant wave height $H_{1/3}$ and the mean zero crossing period is 0.92 times the characteristic period.

The time duration of the stationarity of the i-th state, Δt_i, is assumed to be independent of the i-th state, i.e., $\Delta t_i = \Delta t$ for all states. The total number of stationary periods in N years, N_T, is taken by

$$N_T = 3600 \text{ seconds} * 24 \text{ hrs} * 365 \text{ days} * N \text{ years} / \Delta t$$

and then n can be given by

$$n_i = \frac{(\text{ PPT of each cell})}{1000} * N_T$$

In the actual calculation, Δt is an hour, that is, 3,600 seconds. However, the magnitude of Δt does not affect on the calculation of $\Sigma\alpha_i \Delta t_i n_i$ since the Δt term is cancelled out in the multiplication of Δt and n_i. If we have more information about Δt_i in relation to the sea state, it_i will provide more accurate estimate for $\Sigma\alpha_i \Delta t\, n_i$.

The calculation was performed by PRIME 9650. The probability of exceedance and the PDF are plotted in Fig. 8 and Fig. 9, respectively.

The largest value of wave height which has 1% chance of being exceeded and the most probable value based on the largest value distribution method are tabulated in Table 1. The BMT results, i.e., the most probable value based on the order-statistics and the corresponding probability of exceedance according to the largest value distribution are also given in Table 1. It shows that the BMT result is not consistent with the design periods in terms of the probability of exceedance obtained by the LVD method. For shorter design period, it seems relatively less conservative and for longer period it looks too much conservative. That is, the most probable value 17.96 meters for 1 year has 73% chance of being exceeded while 28.06 meters for 100 years has only 0.1%.

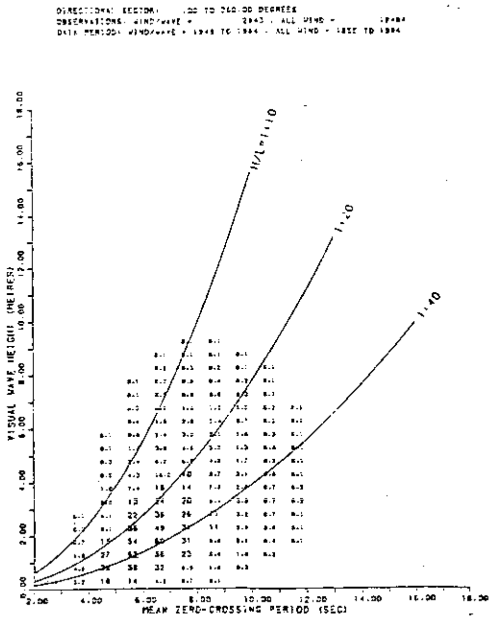

Fig.6 Typical wave scatter data

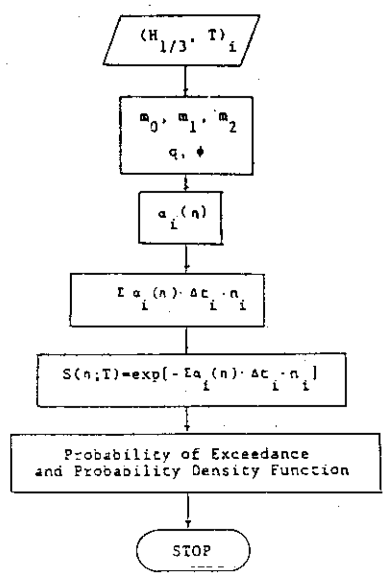

Fig.7 Overall calculation procedure

The "adequate" level of the probability of exceedance in selecting the design wave height, remains a subject of great debate between certification and classification authorities, designers, operators and owners. This may be deduced from the experiences accumulated to date through the structures in successful performance. However, once we set the adequate level of it, we can obtain the design wave heights for various design periods with the consistency in the probability of exceedance.

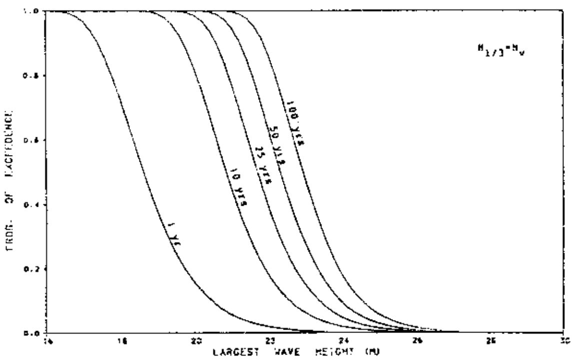

Fig.8 Probability of exceedance of the largest value of wave height

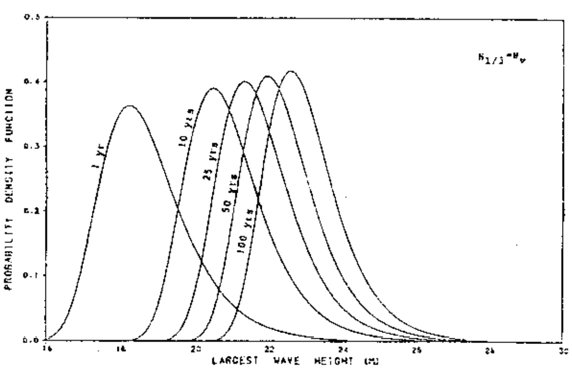

Fig.9 Probability density function of the largest value of wave height

Design Years	Wave Height in Meters			
	LVD Method		BMT Method	
	With 1% of Prob of Exceedance	Most Probable Value	Most Probable Value	Probability of Exceedance (%)
1	22.6	18.3	17.96	73
10	24.5	20.5	23.08	5.5
25	25.3	21.3	25.08	1.3
50	25.8	22.0	26.58	0.4
100	26.4	22.6	28.06	0.1

Table 1 Calculation results of wave heights

CONCLUSIONS

The following summarizes some conclusions:

1) With respect to the largest value distribution during the given period, a new limiting decay rate is proposed assuming a zero-mean Gaussian process. It is less conservative in predicting the largest value than Vanmarcke's which has been found to overpredict the simulation results to the conservative side, especially for the lower region of the threshold.

2) As a practical design purpose, the LVD method is considered as a new tool which determines the design wave height with the acceptable level of the probability of exceedance.

3) For typical wave data, the LVD method is compared with the conventional method based on Weibull distribution fit and order statistics. The LVD method is found to have the following advantages:

a) With the implementation of the LVD method, it is not necessary to assume Weibull distribution for long-term wave heights to find the extreme sea state. We can obtain the long-term LVD of wave height directly from the long-term wave data by evaluating the limiting decay rates and summing them up over all the sea states.

Accordingly, it is not necessary to risk an inaccuracy of parameter estimation and further extrapolate Weibull distribution for the region where data is lacking. In fact, for typical wave data, Weibull distribution inaccurately predicts larger wave heights.

b) In addition, an unjustified use of the North Sea & North Atlantic mean wave period T = 15 sec can be removed in calculating the number of waves during the extreme sea state.

c) The LVD method provides a consistent design wave height in a probabilistic manner. We can control the design wave heights for different design years, keeping the consistent level of the probability of exceedance. This will ease the implementation of reliability theory.

d) In a practical point of view, the LVD method based on upcrossing analysis is still simple, straightforward, and consistent in determining the design wave heights for the long-term design periods. Consequently it can successfully replace the conventional method of determining the design wave height.

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